Thm 3 (Montel Theorem) Let $\mathcal{F}$ be a family of analytic functions on D which is uniformly bounded on compact subsets, then $\mathcal{F}$ is normal.

Pf: Let $K_n = \{ z \in D : |z| \le n \text{ and } \operatorname{dist}(z, \partial D) \ge \frac{1}{n} \}$.
$(n = 1, 2, 3, \cdots)$

Then K_n are compact subsets of D such that

$$K_1 \subset \cdots \subset K_n \subset K_{n+1} \subset \cdots \subset D,$$

and $\bigcup_{n=1}^{\infty} K_n = D$ (Since D is open.)

Let $\{f_j\}$ be a sequence in $\mathcal{F}$.

Then by assumption, $\forall n = 1, 2, 3, \cdots$, $\exists M_n > 0$ such that

$$|f_j(z)| \le M_n, \quad \forall z \in K_n.$$

$\therefore$ $\{f_j\}$ is equibounded on K_n as a family of cts. functions.

Since $\operatorname{dist}(z, \partial D) \ge \frac{1}{n}$, $\forall z \in K_n$,

the disk $\{ |\zeta - z| < \frac{1}{2n} \} \subset K_{2n} \overset{\subset D}{\ } \text{cpt}$.

In fact $\forall \zeta \in \{ |\zeta - z| < \frac{1}{2n} \}$, we have

$$|\zeta| \le |z| + |\zeta - z| < n + \frac{1}{2n} < 2n, \text{ and}$$

$$|\zeta-\eta| \geq |z-\eta| - |\zeta - z| > \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}, \quad \forall \eta \in \partial D.$$

$\therefore \quad \zeta \in K_{2n}$.

Hence f_j are analytic on $\{|\zeta - z| \leq \frac{1}{2n}\}$ and

$$|f_j(\zeta)| \leq M_{2n}, \quad \forall \zeta \in \{|\zeta - z| \leq \tfrac{1}{2n}\}.$$

Cauchy Integral Formula $\Rightarrow$

$$f_j'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = \frac{1}{2n}} \frac{f_j(\zeta)}{(\zeta - z)^2} d\zeta$$

$$\Rightarrow \quad |f_j'(z)| \leq \frac{1}{2\pi} M_{2n} \frac{1}{\left(\frac{1}{2n}\right)^2} 2\pi \cdot \frac{1}{2n}$$

$$= 2n M_{2n}$$

So we've proved that $\forall f_j$, $\forall z \in K_n$

$$|f_j'(z)| \leq 2n M_{2n}.$$

Note that if $z, w \in K_n$ with $|z-w| < \frac{1}{2n}$, then

Then the line segment L joining z, w lies in $K_{\frac{1}{2n}}$.

Applying the above to K_{2n} (instead of K_n), we have

$$|f_j'(\zeta)| \leq 2(2n) M_{2(2n)} = 4n M_{4n}, \quad \forall \zeta \in L.$$

$\therefore$ Integrating along $L \Rightarrow$

$$|f_j(z)-f_j(w)| \le \sup_{\zeta\in L}|f_j'(\zeta)|\,|z-w| \le \left(4nM_{4n}\right)|z-w|.$$

Hence $\forall \varepsilon>0$, let $\delta=\min\{\frac{1}{2n}, \frac{\varepsilon}{4nM_{4n}}\}>0$,

then $\forall f_j$ & $z,w\in K_n$ with $|z-w|<\delta$, we have

$$|f_j(z)-f_j(w)|<\varepsilon.$$

$\therefore$ $\{f_j\}$ is equicontinuous on K_n.

Starting from $n=1$, we apply Arzela-Ascoli Theorem to K_1 and find a subsequence of $\{f_j\}_{j=1}^{\infty}$ converges uniformly on K_1.

Let denote this sequence by $\{f_j^1\}_{j=1}^{\infty}$.

Note that $\{f_j^1\}$ is a subseq. of $\{f_j\}$.

Hence $\{f_j^1\}$ is also uniformly bounded on compact subsets.

Repeating the same argument, we can find a subseq. $\{f_j^2\}$ of $\{f_j^1\}$ such that $\{f_j^2\}$ converges uniformly on K_2. $\therefore$ $\{f_j^2\}$ is (also) a subseq. of $\{f_j\}$

such that $\{f_j^2\}$ converges uniformly on $K_2 \supset K_1$.

By repeating the same argument, we can find, $\forall n = 1, 2, \cdots$

a subseq. $\{f_j^n\}$ of $\{f_j\}$ such that

(1) $\{f_j^{n+1}\}$ is a subseq. of $\{f_j^n\}$, and

(2) $\{f_j^n\}$ converges uniformly on $K_n \supset K_{n-1} \supset \cdots \supset K_1$.

$$
\begin{array}{cccccc}
f_1^1 & f_2^1 & f_3^1 & \cdots & \text{converges uniformly on} & K_1 \\
 & & & & & \cap \\
f_1^2 & f_2^2 & f_3^2 & \cdots & \text{''} \quad \text{''} \quad \text{''} & K_2 \\
 & & & & & \cap \\
f_1^3 & f_2^3 & f_3^3 & \cdots & \text{''} \quad \text{''} \quad \text{''} & K_3 \\
\vdots & \vdots & \vdots & & & \vdots \\
 & & & & & \cap \\
f_1^n & f_2^n & f_3^n & \cdots & \text{''} \quad \text{''} \quad \text{''} & K_n \\
\vdots & \vdots & \vdots & & & \vdots
\end{array}
$$

(Diagonal Trick) Then the seq

$\{f_1^1, f_2^2, f_3^3, \cdots, f_n^n, \cdots\}$ is a subseq. of $\{f_j\}$

and for each $n = 1, 2, \cdots$, $\{f_n^n, f_{n+1}^{n+1}, \cdots\}$ is a subseq. of $\{f_j^n\}$, hence uniformly converges on K_n.

$\Rightarrow \{f_1^1, f_2^2, f_3^3 \cdots\}$ converges uniformly on any K_n.

Since $\bigcup_{n=1}^{\infty} K_n = D$ & $K_n \subset K_{n+1}$,

$\forall$ cpt subset $K \subset D$, $\exists\, n_0$ s.t. $K \subset K_{n_0} \subset D$.

$\therefore \{f_1^1, f_2^2, f_3^3, \cdots\}$ converges uniformly on K.

This completes the proof of the thm. ✗

§6.5 Riemann Mapping Theorem

Thm 1 (Riemann Mapping Theorem) Suppose D is a non-empty simply-connected domain in $\mathbb{C}$ which is not the whole $\mathbb{C}$. If $z_0 \in D$, then there exists a unique conformal map $f: D \to \{|z|<1\}$ such that $f(z_0)=0$ and $f'(z_0)>0$.

Pf: Step 1 : $\exists\, g: D \to \{|z|<1\}$ such that g is conformal onto an open subset of $\{|z|<1\}$ containing $g(z_0)=0$.

Pf of Step 1: Since $D \neq \mathbb{C}$, $\exists\, \alpha \in \mathbb{C}\setminus D$. Then α is not enclosed by any closed contour γ in D (as D is simply-connected.)

Cauchy Thm $\Rightarrow$ $\displaystyle\int_\gamma \frac{dz}{z-\alpha}=0$, $\forall$ such γ.

Hence $h(z) = h_0 + \displaystyle\int_{z_0}^{z} \frac{dz}{z-\alpha}$ is a well-defined analytic function on D, where $h_0 \in \mathbb{C}$ satisfies

$e^{h_0} = z_0 - \alpha$.

Claim 1: $e^{h(z)} = z - \alpha, \ \forall z \in D$.

Pf of claim: $\left(\frac{e^{h(z)}}{z-\alpha}\right)' = \frac{e^{h(z)}}{z-\alpha}\left(h'(z) - \frac{1}{z-\alpha}\right) = 0$.

$\Rightarrow \quad e^{h(z)} = C(z-\alpha)$ for some constant C.

By $z_0 - \alpha = e^{h_0} = e^{h(z_0)} = C(z_0 - \alpha)$,

$\Rightarrow \ C = 1$. ※

Claim 2: $\exists \ \varepsilon > 0$ such that
$$|h(z) - (h(z_0) + 2\pi i)| \geq \varepsilon, \ \forall z \in D.$$

Pf of Claim 2: Suppose not, then for any $\frac{1}{n} > 0$, $\exists z_n \in D$

s.t. $|h(z_n) - (h(z_0) + 2\pi i)| < \frac{1}{n}$

$\Rightarrow \quad h(z_n) \to h(z_0) + 2\pi i$.

By claim 1 and $e^{h(z_n)} \to e^{h(z_0) + 2\pi i}$,

$z_n - \alpha \longrightarrow z_0 - \alpha \Rightarrow z_n \to z_0$.

But then $h(z_n) \to h(z_0) \Rightarrow$
$$h(z_n) - (h(z_0) + 2\pi i) \to 2\pi i \neq 0$$

Contradiction. ※

Claim 3 : $\exists A > 0$ such that

$$g(z) = A\left(\frac{1}{h(z) - h(z_0) - 2\pi i} + \frac{1}{2\pi i}\right)$$

is the required conformal map onto an open subset of $\{|z|<1\}$ containing $0 = g(z_0)$.

Pf of Claim 3 : By claim 2, $h(z) - h(z_0) - 2\pi i \neq 0$, $\forall z \in D$.

$\therefore$ g is analytic.

If $\exists z_1, z_2 \in D$ such that $g(z_1) = g(z_2)$. Then

$$h(z_1) = h(z_2)$$

By claim 1, $z_1 - \alpha = e^{h(z_1)} = e^{h(z_2)} = z_2 - \alpha$

$\Rightarrow$ $z_1 = z_2$ $\Rightarrow$ g is (globally) 1-1.

$\therefore$ g is conformal.

Clearly, $g(z_0) = 0$;

Open mapping Thm $\Rightarrow$ $g(D)$ is open.

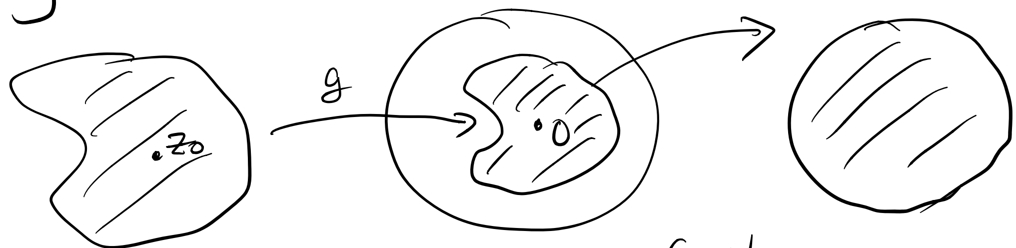
Finally, by claim 2, $|h(z) - h(z_0) - 2\pi i| \geq \varepsilon$, $\forall z \in D$

$\therefore$ $|g(z)| \leq A\left(\frac{1}{\varepsilon} + \frac{1}{2\pi}\right)$

Hence by taking $A = \dfrac{1}{1 + \left(\frac{1}{\varepsilon} + \frac{1}{2\pi}\right)}$, we have $|g(z)| < 1$.

$\#$

Step 2: Note that Step 1 allows us to reduce our problem to $\underbrace{\text{simply-connected}}_{\text{(non-empty)}}$ domain D in $\{|z|<1\}$ containing $z=0$.



And for such D, we consider the family

$$\mathcal{F} = \{ f : D \to \{|z|<1\}, \text{ analytic, } 1\text{-}1, \ f(0)=0 \}.$$

Note that

(1) $\mathcal{F}$ is nonempty ("$f(z)=z$" $\in \mathcal{F}$)

(2) $\mathcal{F}$ is uniformly bounded ($|f(z)|<1, \forall f \in \mathcal{F}$)

By Montel's Thm, $\mathcal{F}$ is a normal family.

<u>Claim</u>: $\exists f_0 \in \mathcal{F}$ such that $|f_0'(0)| = \sup\limits_{f\in\mathcal{F}} |f'(0)|$

<u>Pf of Claim</u>: First note that Cauchy Integral Formula $\Rightarrow \{|f'(0)| : f \in \mathcal{F}\}$ is a bounded set as $|f(z)|<1$ $\forall f \in \mathcal{F}$. Hence $\sup\limits_{f\in\mathcal{F}} |f'(0)| < +\infty$.

Also, $\mathcal{F}$ contains the identity function $f(z)=z$, therefore $\sup_{f\in\mathcal{F}} |f'(0)| \geq 1$ (as $z'=1$).

Now by definition of "sup", $\exists f_n \in \mathcal{F}$ such that

$$|f_n'(0)| \to \sup_{f\in\mathcal{F}} |f'(0)|.$$

As $\mathcal{F}$ is normal, $\exists$ a subseq. $\{f_{n_k}\}$ of $\{f_n\}$ converges uniformly on compact subset of D to a function f_0. By Weierstrass thm, f_0 is analytic, $f_0(0)=0$ and $f_{n_k}'(0) \to f_0'(0)$ as $k\to\infty$.

In particular, $$|f_0'(0)| = \sup_{f\in\mathcal{F}} |f'(0)| \geq 1.$$

$\Rightarrow f_0$ is non-constant.

Then by Hurwitz thm of f_{n_k} 1-1 $\forall k$ implies f_0 is also 1-1. (Ex!)

Finally, by uniform convergence and continuity, we must have

$$|f_0(z)| \leq 1, \ \forall z \in D.$$

Then maximum modulus principle implies

$$|f_0(z)| < 1, \quad \forall z \in D$$

as f_0 is non-constant. Therefore, $f_0 \in \mathcal{F}$ and satisfies $|f_0'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|$. ※

Step 3: $f(z) = e^{-i \arg f_0'(0)} f_0(z)$ is the required conformal map onto $\{|z|<1\}$.

(i.e. $f : D \to \{|z|<1\}$ conformal $f(0)=0$, $f'(0)>0$)

(Then together with Step 1, $f \circ g$ is the required map up to a rotation.)

Pf of Step 3: It remains to show that $f_0(z)$ is surjective.

Suppose not, then $\exists\, \alpha \in \{|z|<1\} \setminus f_0(D)$.

$$\Rightarrow \quad f_0(z) \neq \alpha, \quad \forall z \in D.$$

Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$. $\quad$ (pole $\left|\frac{1}{\bar{\alpha}}\right| > 1$)

Then $\psi_\alpha \circ f_0$ is analytic, 1-1, and

$$\psi_\alpha \circ f_0(z) \neq 0, \quad \forall z \in D$$

Since D is simply-connected & $\psi_0 \circ f_0$ is 1-1, cts,

$\psi_\alpha \circ f_0(D)$ is also simply-connected. Therefore,

by the same argument as in Step 1,

$\varphi(w) = e^{\frac{1}{2}\log w}$ is well-defined on $\psi_\alpha \circ f_0(D)$.

(Ex!) Consider

$$f_1(z) = \psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha \circ f_0(z), \quad z \in D$$

$(\alpha = \psi_\alpha \circ f_0(0) \in \psi_\alpha \circ f_0(D))$

$\Downarrow$

$\varphi(\alpha)$ well-defined.

where $\psi_{\varphi(\alpha)} = \dfrac{\varphi(\alpha) - z}{1 - \overline{\varphi(\alpha)}\, z}$

$|\alpha| < 1 \Rightarrow |\varphi(\alpha)| < 1$.

Clearly, $f_1(z)$ is analytic.

$$f_1(0) = \psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha \circ f_0(0)$$
$$= \psi_{\varphi(\alpha)} \circ \varphi(\alpha) = 0.$$

Note that by Thm 2 of §6.3, ψ_α and $\psi_{\varphi(\alpha)}$ map $\{|z|<1\}$ onto $\{|z|<1\}$. Also $|f_0(z)| < 1$ and

$$|\varphi(w)| = |e^{\frac{1}{2}\log w}| = e^{\frac{1}{2}\log|w|} < 1 \quad \text{for } |w| < 1.$$

$\therefore \ |f_1(z)| < 1, \ \forall z \in D$

Moreover, "$f_1(z_1) = f_1(z_2)$" $\Rightarrow$ "$\varphi \circ \psi_\alpha \circ f_0(z_1) = \varphi \circ \psi_\alpha \circ f_0(z_2)$"

as $\psi_{\varphi(\alpha)}(z)$ is invertible.

Taking square, $\quad \psi_\alpha \circ f_0(z_1) = \psi_\alpha \circ f_0(z_2)$.

Hence $z_1 = z_2$ as ψ_α & f_0 are 1-1.

$\therefore$ f_1 is 1-1.

Altogether, we have $f_1 \in \mathcal{F}$.

However, if we rewrite $f_1 = \psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha \circ f_0$

by using the square mapping $\zeta(w) = w^2$, then

$$f_0 = \psi_\alpha^{-1} \circ \zeta \circ \psi_{\varphi(\alpha)}^{-1} \circ f_1$$

$$= \Phi \circ f_1 \quad \text{where } \Phi = \psi_\alpha^{-1} \circ \zeta \circ \psi_{\varphi(\alpha)}^{-1}.$$

Note that $|\Phi(z)| < 1, \ \forall z \in \{|z| < 1\}$ and

$\Phi(0) = 0 \quad (\text{as } 0 = f_0(0) = \Phi(f_1(0)) = \Phi(0))$

but Φ is not 1-1 (as ζ is not 1-1)

Hence equality case cannot be true in the Schwarz

Lemma. $\therefore$ $|\Phi'(0)| < 1$.

$$\Rightarrow \quad |f_0'(0)| = |\Phi'(f_1(0))\, f_1'(0)|$$
$$= |\Phi'(0)|\,|f_1'(0)| < |f_1'(0)|$$

Contradicting $|f_1'(0)| \le |f_0'(0)|$.

$\therefore$ f_0 must be surjective.

This completes the proof of the existence part of the Riemann Mapping Thm.

<u>Step 4</u> (Uniqueness)

Let f, g be 2 such mappings. Then

$g \circ f^{-1} : \{|z|<1\} \to \{|z|<1\}$ conformal,

satisfying $g \circ f^{-1}(0) = 0$ & $(g \circ f^{-1})'(0) = \dfrac{g'(z_0)}{f'(z_0)} > 0$.

$\therefore$ Thm 2 of §6.3 $\Rightarrow$ $g \circ f^{-1}(z) = z$. ##

This completes the proof of Riemann Mapping Theorem.